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Strictly positive fragments of modal logics

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Relevance of the research topic and the degree of its development

This work is devoted to the study of strictly positive fragments of modal logics. Such fragments were considered earlier in the context of universal algebra, in the study of description logics, and in provability logic.

Strictly positive modal formulas are constructed from variables and the constant $\top$ using connectives $\wedge$ and $\Diamond$. Accordingly, from the point of view of universal algebra, implications between strictly positive formulas (hereinafter *sequents*) correspond to identities in the language of lower semilattices with several monotone operators. Strictly positive logics correspond to varieties of such algebras.

One of the first works in this direction was the work of M. Jackson [1], in which semilattices with closure semilattices (CSL) were considered and the lattice of subvarieties of normal CSLs was described, that is, the lattice of extensions of strictly positive logic $\mathcal{SP}(S5)$. It was also shown there that finite CSL varieties with identity have a finite basis.

Strictly positive modal logics play an important role in research on the theory of ontological databases and descriptive logic. Ontological databases are databases equipped with some, necessarily limited, ability to draw logical conclusions based on available facts. To achieve computational efficiency, ontological databases use specialized languages (such as OWL), which are built based on so-called *description logics*. From a theoretical point of view, description logics can be considered as variants of modal logics.

Due to the relevance of the considered problems, the study of description logics and related problems is currently an actively developing and important applied area of research.

One of the important problems in the field of description logics is the construction of logical languages that would be both sufficiently expressive and efficient when used in ontological databases. Regarding this, the language of strictly positive modal logic represents a convenient compromise between expressiveness and efficiency.

Thus, the medical terminology database SNOMED CT uses strictly positive logic $\mathcal{EL}$, described in the works [2, 3, 4], together with an effective resolution algorithm. The $\mathcal{EL}$ logic allows one to answer a query in polynomial time depending on the length of the query and the size of the database, as shown in the work [2]. For many description logics, algorithms were known that run in exponential time in the worst case, but are quite applicable in practice thanks to optimizations. The work [4] shows that the deciding algorithm for logic $\mathcal{EL}$ is not inferior to such algorithms in performance.

Another source of interest in strictly positive modal logics is given by the research in the field of provability logic. Currently, provability logics are actively used to analyze the properties of formal axiomatic theories [5], including the study of their ordinal characteristics and the construction of canonical systems of ordinal notation. As it turns out, for many applications in proof theory it is sufficient to use the language of strictly positive provability logic, which leads to simpler systems than systems based on the full language of modal logic.

Thus, the multimodal provability logic GLP, which is actively used in this area, is Kripke-incomplete and PSPACE-complete [7], while its strictly positive fragment RC is both Kripke-complete and polynomially solvable [8]. The closed fragment of logic RC, as shown in [5], is a natural ordinal notation system for the characteristic ordinal of Peano arithmetic ε_0. This system is used in the analysis of formal arithmetic and its fragments based on the methods of provability logic.

Next, in the studies of L.D. Beklemishev [9, 10, 11] and other authors [12, 13, 14, 15, 16] new arithmetic interpretations of strictly positive logics were proposed — for example, for logics RC and $RC\omega$ the modalities correspond to uniform reflection principles. The logic $RC\omega$ is also polynomially solvable and complete with respect to a relatively simple class of finite Kripke models. In general, we can say that the apparatus of strictly positive logic has occupied an important place in research on provability logic and its applications to the ordinal analysis of logical theories.

This paper addresses general theoretical questions about strictly positive logics, which are not related to specific applications of these logics in descriptive logic or proof theory. However, these can help in understanding the specifics of strictly positive logics in each of these areas of application. Such general questions of strictly positive logics were considered in several works by M. Zakharyashchev, F. Voltaire, A. Kurush, S. Kikot, and others [17, 18, 19, 20, 21, 22, 23].

Particularly important is the recent work by S. Kikot et al. [22], which considers the question of Kripke completeness for strictly positive logics. It is known that strictly positive logic is Kripke complete if and only if it is a strictly positive fragment of some normal modal logic. In contrast to modal logics that are not Kripke complete, there are quite simple examples of strictly positive logics that are not Kripke complete — for example, the logic generated by the axiom $\Diamond p \to \Diamond q$. In the paper [22] it is proved that all the extensions of the modal logic $S5$, with two exceptions, are Kripke complete; it is also shown that the Kripke completeness problem is undecidable. The inverse question is also considered: is it possible to define a given class of Kripke frames using only formulas of strictly positive logic? The necessary condition for this and examples of non-definable (in this sense) classes of frames are found. In particular, the class of linear frames characterizing the logic $K4.3$ is non-definable.

Purpose of work and main tasks In this paper, we study the following questions about strictly positive logics.

Axiomatization of strictly positive fragments of modal logics. The question of a convenient strictly positive axiomatization of fragments of standard modal logics is important for applications in provability logic since we usually establish the correctness of an arithmetic interpretation by induction on the length of the formal derivation. The presence of an axiomatization convenient for analysis also makes it possible to apply syntactic methods to the study of strictly positive logics.

Standard logics, for strictly positive fragments of which no convenient axiomatization was known before the author's work, include logic $\mathrm{S5}_n$ (corresponding to epistemic logic with n agents) and logic K4.3 (the logic of transitive linear frames). In this paper, we find such an axiomatization and prove the solvability of strictly positive fragments of these logics in polynomial time.

Study of modal companions of the strictly positive fragment of logic K4.

We consider the following connection between modal logics and strictly positive logics [24]. Modal logic L is associated with its strictly positive fragment $\mathcal{SP}(L)$, which is defined as the set of all sequents (implications between strictly positive formulas) from L. Given a strictly positive logic P, one can obtain a normal modal logic $K \oplus P$ — closure of $K \cup P$ with respect to the inference rules of modus ponens, substitution and amplification (notation $\mathcal{ML}(P)$). When $P = \mathcal{SP}(L)$, we call L the *modal companion* of P. The maps $\mathcal{SP}$ and $\mathcal{ML}$ together form a Galois correspondence, analogous to the correspondence between modal and superintuitionistic logics.

For superintuitionistic logics this correspondence has been studied in detail, starting with [25]. It is known that every superintuitionistic logic has its greatest modal companion, and that the corresponding maps are isomorphisms of the lattices of extensions of the intuitionistic logic IPC and normal extensions of the Grzegorczyk logic Grz (Blok–Esakia theorem, see [26]). The question of whether the greatest modal companion for strictly positive logics exists was posed by Beklemishev in [24], in particular, for the strictly positive fragment of logic $K4$. We answer this question in the negative and show that the strictly positive logic $\mathcal{SP}(K4)$ has at least two different maximal modal companions. Until now, no examples of normal strictly positive modal logics with more than one modal companion were known.

Well-know examples include the map $\mathcal{ML}_{S4.3}(P) = S4.3 \oplus P$, which is studied in [21], or the map $\mathcal{ML}_{S5}(P) = S5 \oplus P$ from the work [1]. Moreover, any strictly positive logic extending $\mathcal{SP}(S4.3)$ has the greatest modal companion [21, Theorems 5.2, 5.3]. The same is true for strictly positive logics extending $\mathcal{SP}(S5)$ — on the one hand, simply because $S4.3 \subset S5$, on the other hand, this fact can be established separately, since all strictly positive logics extending $\mathcal{SP}(S5)$ and all normal extensions of the logic $S5$ [27] are known.

In this paper, we analyze extensions of the logic $K4$ using for this purpose the canonical frame formulas introduced by M. Zakharyashchev (see [28, Section 9]). Although we do not obtain a complete description of all modal companions of the logic $\mathcal{SP}(K4)$, we nevertheless describe the set of all modal companions of $\mathcal{SP}(K4)$ in the class of extensions $K4$ by canonical formulas of irreflexive frames, including the greatest logic in this set. We also find a criterion for whether some modal logic is a modal companion of $\mathcal{SP}(K4)$ and show that the Gödel-Löb logic GL is not a maximal modal companion of $\mathcal{SP}(K4)$. This answers another question posed by L.D. Beklemishev [24].

The implication relation on strictly positive formulas as a well-quasi order. The work also considers the implication relation on strictly positive formulas as a quasiorder. The main result of this section of the dissertation is that in $K4$ logic this relation, limited to formulas of a fixed finite number of variables, is a well-quasi order, that is, it contains neither infinite decreasing chains nor infinite antichains.

The theory of well-quasi orders is well-known and interesting from the point of view of set theory and combinatorics, since many natural structures, such as order on words and order on trees, the relation «to be a minor» in graphs, are well-quasi orders. One of the most important such structures, as shown in the article [31], is a well-quasi order by embedding on linear orders.

Well-quasi orders are also actively used to prove the termination of term rewriting systems (for example, as in the article [32]) and are used to study the decidability problem of fragments of predicate logic (see [33]). Applications of well-quasi orders in proof theory are also known — for example, the finite form of Kruskal's theorem provides an example of a natural combinatorial statement that is not provable in the strong arithmetic theory ATR_0 [34]. A more detailed logical analysis of Kruskal's theorem, together with its finite form and the derivation of the ordinal type of a well-quasi order on trees, is given in the article [35]. The article [36] discusses the correspondence between another combinatorial statement independent of Peano arithmetic — Worm's principle — and the well-known well-quasi order on words of natural numbers or, which is the same, on closed modal formulas of multimodal logic of provability GLP^-. This well-quasi order has a similar nature to the one studied in this work.

Recent results on well-quasi orders are collected in the book [37], in particular, in the article [38] various ordinal characteristics of well-quasi orders, including their ordinal type, are considered. We find upper and lower bounds for the ordinal type for the studied well-quasi order on strictly positive formulas, but there is currently a gap between these bounds.

Scientific novelty

All results presented in the dissertation are new.

Theoretical and practical significance of the work

The dissertation is theoretical. The results obtained in it are important for modal logic, as well as for the theory of well-quasi orders.

Research methods

The thesis uses the canonical model method, the method of adding vertices to the model, tree linearization, characteristic formulas and other classical methods of modal logic. Kruskal's theorem and the quasi-embedding lemma from the theory of well-quasi orders are also used.

The main defense points

1. A finite axiomatization and a result on the polynomial solvability of a strictly positive fragment of logic $K4.3$ are obtained.

2. It has been proven that the natural implication order in $K4$ on strictly positive formulas of a fixed number of variables is a well-quasi order; estimates for its ordinal type are also obtained.

3. A criterion is obtained for whether a modal logic is a modal companion of $K4^+$, and with its help it is established which of the extensions of $K4$ by cofinal formulas are modal companions. In particular, it is proven that $K4^+$ has at least two different maximal modal companions.

Approbation of research

Reports at conferences and seminars:

— International Conference «Workshop on Proof Theory, Modal Logic and Reflection Principles», Russia, 2017.

— International Conference «Workshop on Proof Theory, Modal Logic and Reflection Principles», Spain, 2019.

— Seminar of the Mathematical Logic Department «Provability theory», Moscow Institute of the Academy of Sciences;

— Seminar «Modern problems of mathematical logic», Higher School of Economics.

Publications

The main results on the topic of the dissertation are presented in 3 papers. Two of them were published in journals included in the list of the Higher Attestation Commission; 2 of them — in journals indexed by the Scopus database; 2 of them — in journals indexed by the Web of Science database. All results submitted for defense were obtained by the author of the dissertation independently.

Contents

The introduction substantiates the relevance of the research carried out within the framework of this dissertation and provides an overview of known results associated with various properties of strictly positive fragments.

In Section 2, the following definitions are given:

Modal logic *formulas* are built-up from variables $\mathrm{Var} = \{p_0, p_1, \ldots\}$ and the constant $\top$ using logical connectives $\wedge$, $\neg$ and unary modality $\Diamond$. The symbols $\vee$, $\rightarrow$, $\Box p = \neg\Diamond\neg p$ and $\Box^+ p = p \wedge \Box p$ are considered standard abbreviations.

Normal modal logic is a set of modal formulas, closed with respect to inference rules modus ponens, substitution, and necessitation. The smallest normal modal logic is called K; the symbol $\oplus$ denotes the addition of a formula to the logic and next taking the closure with respect to the mentioned rules. E.g., $K \oplus (\Box p \rightarrow \Box\Box p)$ is the logic $K4$.

Strictly positive formulas are the modal formulas, built-up using only the connectives $\wedge$ and $\Diamond$.

A *sequent* is a formula of the type $A \rightarrow B$, where A and B are strictly positive formulas.

The *strictly positive fragment* of a modal logic L is the set of all sequents from L (notation $\mathcal{SP}(L)$).

The calculus of strictly positive logic K^+ consists of the following axioms and inference rules:

1. $A \rightarrow A, A \rightarrow \top, \quad \dfrac{A \rightarrow B, \;\; B \rightarrow C}{A \rightarrow C}$

2. $A \wedge B \rightarrow A, \;\; A \wedge B \rightarrow B, \quad \dfrac{A \rightarrow B, A \rightarrow C}{A \rightarrow B \wedge C}$

3. $\dfrac{A \rightarrow B}{\Diamond A \rightarrow \Diamond B}$

In general, a *strictly positive logic* is a set of sequents, closed with respect to the inference rules listed above. This corresponds to the semantics of lower semilattices with monotone operators (SLO, for short): a sequent is derivable in K^+ if and only if it is true in every SLO. By $K4^+$ we denote the calculus K^+ together with the axiom $\Diamond\Diamond A \to \Diamond A$.

Next, we define standard Kripke semantics:

- A *Kripke frame* is a pair $\mathcal{F} = (W, R)$, where W is a set, and R is a binary relation on W.

- A *Kripke model* is a triple $\mathcal{M} = (W, R, V)$, where (W, R) is a Kripke frame, and V, or *valuation* is a map $V\colon \mathrm{Var} \to 2^W$.

- The valuation on variables $V(p)$ determines the valuation on formulas $V(\varphi)$. The valuation is defined by induction on φ: $V(\top) = W$, $V(\neg\varphi) = W \setminus V(\varphi)$, $V(\varphi_1 \wedge \varphi_2) = V(\varphi_1) \cap V(\varphi_2)$, $V(\Diamond\varphi) = \{x \mid \exists y \in V(\varphi)\, xRy\}$.

- We write $\mathcal{M}, x \models \varphi$ if $x \in V(\varphi)$ and $\mathcal{M} \models \varphi$ if $V(\varphi) = W$. We also $\mathcal{F} \models \varphi$ if $\mathcal{M} \models \varphi$ for every model $\mathcal{M}$ based on this frame.

- Let L be a modal logic, and $\mathcal{F}$ — a Kripke frame; we say that $\mathcal{F}$ is a frame for L if $\varphi \in L \Rightarrow \mathcal{F} \models \varphi$ for every modal formula φ.

- Let C be a class of Kripke frames (or models), and L — a modal logic. We say that L is characterized by C (or that L is the logic for class C) if $\varphi \in L \Leftrightarrow \mathcal{F} \models \varphi$ for every modal formula φ and frame/model $\mathcal{F} \in C$. E.g., the logic $K4$ is the logic of all transitive frames.

  Also, in the thesis we discuss the logics $S5 = K4 \oplus (\Box p \to p) \oplus (p \to \Box\Diamond p)$ — the logic of equivalence relations, $K4.3 = K4 \oplus \Box(\Box^+ p \to q) \vee \Box(\Box^+ q \to p)$ — the logic of linear frames and Gödel-Löb logic $GL = K4 \oplus \Box(\Box p \to p) \to \Box p$ — the logic of transitive frames that do not contain infinite ascending chains.

- For denoting the variables that are true in some vertex (in a Kripke model), we use the notation $\mathrm{Var}(x) = \{p \mid x \in V(p)\}$.

Next, we define homomorphism and canonic trees.

Let $\mathcal{M}_1 = (W_1, R_1, V_1)$ and $\mathcal{M}_2 = (W_2, R_2, V_2)$ be some transitive Kripke models. A map $f\colon W_1 \to W_2$ is a *homomorphism* if:

1. $\forall x, y \in W_1\ xR_1y \to f(x)R_2f(y)$;

2. $\forall x \in W_1\ \mathrm{Var}_1(x) \subset \mathrm{Var}_2(f(x))$.

In case both models have roots and $f(r(\mathcal{M}_1)) = r(\mathcal{M}_2)$, we call f a *root-preserving* homomorphism.

For every strictly positive formula A its *canonic tree* $T[A]$ is a tree-like Kripke model, defined by induction as follows.

In case A is a variable or the constant $\top$, $T[A]$ is a singleton validating only A.

In case $A = B \wedge C$, $T[A]$ is made from $T[B]$ and $T[C]$ by joining their roots. A variable is true in the new root, if and only if it is true in either the root of $T[B]$ or the root of $T[C]$.

When $A = \Diamond B$, $T[A]$ is obtained from $T[B]$ by adding a new root, which reaches every vertex of $T[B]$ and invalidates every variable.

Theorem 1. *[8] Let A, B be strictly positive formulas. The following statements are equivalent:*

(i) $K4^+ \vdash (A \to B)$;

(ii) $T[A], r(T[A]) \models B$;

(iii) there exists a root-preserving homomorphism $f\colon T[B] \to T[A]$.

Next, following the book by A. Chagrov and M. Zakharyaschev [28], we define canonical frame formulas $\alpha(\mathcal{F}, \mathcal{D}, \bot)$ and $\alpha(\mathcal{F}, \mathcal{D})$ (here $\mathcal{F}$ is a finite transitive rooted Kripke frame, $\mathcal{D}$ — a subset of antichains in $\mathcal{F}$, excluding reflexive singletons, $\mathcal{D}^{\#}$ — the set of all such antichains in $\mathcal{F}$) and mention how they are related to normal extensions of $K4$.

Theorem. *[28, Theorem 9.39] For every transitive frame $\mathcal{S}$,* *(i)* $\mathcal{S} \not\models \alpha(\mathcal{F}, \mathcal{D}, \bot)$, *if and only if there exists a cofinal reduction from* $\mathcal{S}$ *onto* $\mathcal{F}$*, satisfying the cofinal domain condition (CDC) for every antichain in* $\mathcal{D}$*;*

(ii) $\mathcal{S} \not\models \alpha(\mathcal{F}, \mathcal{D})$, *if and only if there exists a reduction from* $\mathcal{S}$ *onto* $\mathcal{F}$*, satisfying CDC for every antichain in* $\mathcal{D}$*.*

Theorem. *[28, Theorem 9.43] There exists an algorithm, which takes any modal formula* φ *as the input and outputs the set of canonical frame formulas* $\alpha(\mathcal{F}_1, \mathcal{D}_1, \bot), \alpha(\mathcal{F}_2, \mathcal{D}_2, \bot), \ldots, \alpha(\mathcal{F}_n, \mathcal{D}_n, \bot)$*, such that* $K4 \oplus \varphi = K4 \oplus \alpha(\mathcal{F}_1, \mathcal{D}_1, \bot) \oplus \alpha(\mathcal{F}_2, \mathcal{D}_2, \bot) \oplus \ldots \oplus \alpha(\mathcal{F}_n, \mathcal{D}_n, \bot)$*.*

We also define well-quasi order and its ordinal type.

A pair $(\mathscr{A}, \preceq)$, where $\preceq$ is a binary relation on set $\mathscr{A}$, is called a *quasiorder* if $\preceq$ is reflexive and transitive. Similarly, a partial order (or a partially ordered set) is a reflexive, transitive, and antisymmetric relation; by factorizing on the equivalence relation, we can obtain a partially ordered set from every quasiorder.

A quasiorder $(\mathscr{A}, \preceq)$ is called *well-quasiorder*, if for every infinite sequence $a_1, a_2, \ldots, a_n, \ldots$ of elements from $\mathscr{A}$ there exists a pair of indexes $i < j$ such that $a_i \preceq a_j$. We will further use an equivalent definition, namely, every infinite sequence contains an infinite non-decreasing subsequence. A *well-order* we will call an antisymmetric well-quasiorder; it is clear that, if $(\mathscr{A}, \preceq)$ is a well-quasiorder, then $(\mathscr{A}/_{\sim}, \preceq)$ (quotient by the relation $\preceq$) is a well-order.

Given a well-order $(X, \preceq)$, one can consider all the linear orders containing it and find the supremum of corresponding ordinals, we will call such a supremum *ordinal type* and denote as $o(X, \preceq)$. In the work [39] it is shown that this supremum is indeed a maximum (i.e., there exists a linear order containing $\preceq$ and corresponding to the ordinal $o(X, \preceq)$).

The ordinal type of a well-quasiorder $(X, \preceq)$ is the ordinal type of the corresponding well-order $(X/_{\sim}, \preceq)$.

In Section 3 we consider the strictly positive fragment of polymodal logic

$S5_m$, i.e. the logic of m equivalence relations. We find its axiomatization $S5_m^+$ by translating the modal axioms, which correspond to transitivity, reflexivity, and Euclidean relations of Kripke frames, into the strictly positive language. Next, we formulate a

Theorem 2. *A sequent is derivable in $S5_m^+$ if and only if it is derivable in $S5_m$.*

To prove the theorem, we use the canonical model method. We define the model $\mathcal{M} = (W, R_1 \dots R_m, v)$, which consists of all nonempty $S5_m^+$ theories, and show that a sequent is true in this model if and only if it is derivable in $S5_m^+$; as all the relations in $\mathcal{M}$ are equivalence relations, this proves the theorem.

Next, we define an analog of the canonic tree $(\tilde{T})$ for the logic $S5_m$ and prove a statement, which is similar to Theorem 1:

Theorem 3. *Let A, B be some strictly positive formulas. The following statements are equivalent:*

(i) $S5_m^+ \vdash (A \to B)$;

(ii) $\tilde{T}[A], r(A) \models B$;

(iii) there exists a root-preserving homomorphism $f : \tilde{T}[B] \to \tilde{T}[A]$.

As for the derivability checking of the sequent $A \to B$ in the logic $S5_m$ it suffices to check whether B is true in the model $\tilde{T}[A]$, this proves

Corollary. *The strictly positive fragment of the logic $S5_m$ is polynomially decidable.*

In Section 4 we study the strictly positive fragment of $K4.3$, the logic of linear frames. We show that, given a canonic tree, we can obtain all its linearizations by repeatedly applying elementary transformations (motivated by the idea from an unpublished work of S. Kaniskin [40], where a similar thing is proposed for the strictly positive fragment of $K4.3$ with disjunction).

Next we consider the sequent $comwit_2 = \Diamond(p \wedge \Diamond q_1) \wedge \Diamond(p \wedge \Diamond q_2) \to \Diamond(p \wedge \Diamond q_1 \wedge \Diamond q_2)$, which substitutes the (.3) axiom in strictly positive language, define the antichain completion $\mathcal{M}^{\curlyvee}$ of a given Kripke model $\mathcal{M}$ and prove the following theorem:

Theorem 4. *Let A, B be strictly positive formulas. The following statements are equivalent:*

(i) $T[A]^{\curlyvee}, \{r(A)\} \models B$*;*

(ii) $K4^{+} + comwit_2 \vdash A \to B$*;*

(iii) $(A \to B) \in K4.3$*;*

(iv) B *is true at the root of every linearization of* $T[A]$*.*

Finally, we point out that the logic of flower-like frames (which, in difference from linear ones, may contain pairwise incomparable maximal elements) has the same strictly positive fragment as the logic $K4.3$.

We call a relation R *flower-like* if it is transitive and all non-maximal elements are comparable.

Theorem 5. *The strictly positive fragment of the logic $K4.3$ equals to the positive fragment of the logic of flower-like frames.*

In Section 5 we prove polynomial decidability of the strictly positive fragment of $K4.3$. For this purpose, we define an algebraic structure on the Kripke model $\mathcal{M} = (W, R, v)$. Precisely, we define a function g, which, given a strictly positive formula, evaluates some set of subsets of W, and a function f, which, given such a set, outputs a set of antichains in W. By some supplementary lemmas, showing the properties of these functions, we prove the following

Theorem 6. *Let X be a vertex in $\mathcal{M}^{\curlyvee}$ (i.e., an antichain in W), and A is a strictly positive formula. Then $\mathcal{M}^{\curlyvee}, X \models A$, if and only if $X \in f(g(A))$.*

As the functions f, g are polynomially computable, the polynomial decidability of the strictly positive fragment of $K4.3$ follows from Theorems 4, 6.

In Section 6 we consider the set Fm of strictly positive formulas in a fixed finite set of variables and the relation $\preceq$ on it, defined as $A \preceq B$, if in $K4$ $B \to A$ is derived (that is, the inverse of the natural order of implication). We prove the following

Theorem 7. *The relation $\preceq$ is a well-quasiorder.*

Then the definitions of a negative formula, a negative theory, a dual operator $*$ on formulas, and a theory dual to a given strictly positive one are given; we prove

Theorem 8. *Let T be a strictly positive theory on a finite set of variables, closed under the axioms and inference rules of $K4^+$. Then the dual theory T^* has a finite axiomatization.*

Finally, using auxiliary lemmas (including the ordinal type of order on strings [39] and the quasi-embedding lemma [41]), we obtain upper and lower bounds for the ordinal type of the considered well-quasiorder $\preceq$:

Theorem 9. *The ordinal type $(Fm, \preceq)$ on n variables is at least $\omega^{\omega^{2^n-1}}$.*

Theorem 10. *The ordinal type $(Fm, \preceq)$ on n variables is at most $\varepsilon_0 \times 2^n$.*

In Section 7 we introduce the concept of a *base set* for a Kripke model — a set of Kripke models, which is equivalent (in the sense of validating an arbitrary strictly positive formula) to the given Kripke model. With its help, the following criterion for a modal companion is proved:

Theorem 11. *Let L be a normal extension of $K4$, which is characterized by the class of models C. Then L is a modal companion of $K4^+$ if and only if for any strictly positive formula A, there is a set of models $\mathfrak{M}(A)$ that is the base set for $T[A]$ and a subset of C.*

Also, using the properties of a well-quasiorder from the previous section, a stronger version of this criterion is proven:

Theorem 12. *Let L be a normal extension of $K4$, which is characterized by the class of models C. Then L is a modal companion of $K4^+$ if and only if for any strictly positive formula A, there exists a* ***finite*** *set of* ***finite*** *models $\mathfrak{M}(A)$, which is the base model for $T[A]$, and each model in $\mathfrak{M}(A)$ is a submodel of some model in C.*

In Section 8 we use a criterion to determine which logics are (or are not) modal companions of $K4^+$. For convenience, we introduce the following notation: $\mathcal{F}_0$ — the minimal nonlinear frame with a root, $\mathcal{F}_1$ — the minimal non-tree frame with a root, $\mathcal{F}_n^{lin}$ — linear frame from n vertices. We prove

Lemma 1. *The logic $K4 \oplus \alpha(\mathcal{F}_n^{lin}, \mathcal{D}^{\#}, \bot)$ is not a modal companion of $K4^+$.*

Lemma 2. *The logic $K4 \oplus \alpha(\mathcal{F}_0, \emptyset, \bot) \oplus \alpha(\mathcal{F}_1, \emptyset, \bot)$ is a modal companion of $K4^+$.*

Theorem 13. *Let $\mathfrak{F}$ be a set of finite irreflexive transitive rooted frames. Then the logic $K4 \oplus \{\alpha(\mathcal{F}, \mathcal{D}, \bot) \mid \mathcal{F} \in \mathfrak{F}\}$ (with $\mathcal{D}$ being some arbitrary sets of antichains) is a modal companion of $K4^+$, if and only if $\mathfrak{F}$ does not contain a linear frame.*

The next two lemmas show that GL is not a maximal modal companion:

Lemma 3. *Let $\mathfrak{F}$ be the set of all non-linear frames with height k. Then the logic $GL \oplus \{\alpha(\mathcal{F}, \mathcal{D}^{\#}, \bot) \mid \mathcal{F} \in \mathfrak{F}\}$ is a modal companion of $K4^+$.*

Let $\mathfrak{F}_k$ denote the set of all non-linear frames with height at most k, and $L_k = GL \oplus \{\alpha(\mathcal{F}, \mathcal{D}^{\#}, \bot) \mid \mathcal{F} \in \mathfrak{F}_k\}$.

Lemma 4. *The logic $\bigcup\limits_{k=1}^{\infty} L_k$ is a modal companion of $K4^+$.*

Lemma 5. *The logic $GL \oplus \alpha(\mathcal{F}_0, \emptyset, \bot)$ is not a modal companion of $K4^+$.*

As (separatedly) the logics GL and $K4 \oplus \alpha(\mathcal{F}_0, \emptyset, \bot)$ are modal companions of $K4^+$, we conclude

Theorem 14. *The strictly positive logic $K4^+$ has at least two different maximal modal companions.*

In conclusion, the main results of the work are presented, which are as follows:

1. A finite axiomatization and a result on the polynomial solvability of a strictly positive fragment for the logic $K4.3$ are obtained.

2. It has been proven that the natural sequence order in $K4$ on strictly positive formulas of a fixed number of variables is a well-quasi order. We also provide lower and upper bounds of its ordinal type.

3. A criterion is obtained for whether a modal logic is a modal companion of $K4^+$, and with its help it is established which of the extensions of $K4$ by cofinal formulas are modal companions. In particular, it is proven that $K4^+$ has at least two different maximal modal companions.

Several directions for further research are also indicated. First, it is interesting to further analyze extensions of $K4$ to see if they are modal companions of $K4^+$, including getting more examples of modal formulas that have non-trivial consequences in the form of sequents (as shown in Lemma refnotcomp). Secondly, it would be interesting to more accurately estimate the ordinal type of a well-quasiorder on strictly positive formulas in n variables. From the bounds we obtained, it is unclear whether this ordinal type depends on the parameter n. The third possible direction is the study of strictly positive fragments of other modal logics, in addition to the considered $S5_m$ and $K4.3$; for example, for each logic from this work that is not a modal companion of $K4^+$, we indicated the specific sequent that this logic contains. Starting from this sequent, one can find an axiomatization of a strictly positive fragment of one of these logics.

Список литературы

[1] M. Jackson 'Semilattices with closure', *Algebra Universalis* 52:1–37, 2004.

[2] *Baader F., Brandt S., Lutz C.* Pushing the EL envelope. // Proc. of the 19th Joint International Conference of Artificial Intelligence. — 2005.

[3] *Baader F., Brandt S., Lutz C.* Pushing the EL envelope. // LTCS-Report 05-01, Institute for Theoretical Computer Science, Dresden University of Technology. — 2005.

[4] *Baader F., Brandt S., Suntisrivaraporn* Is tractable reasoning in extensions of the description logic EL useful in practice? // Proc. of the Methods for Modalities Workshop, Berlin, Germany. — 2005.

[5] Beklemishev L. D., Reflection principles and provability algebras in formal arithmetic., Russian Math. Surveys, 60:2 (2005), 197–268.

[6] *Boolos G.* The Logic of Provability. // Cambridge University Press. — 1993.

[7] *Shapirovskiy I.* PSPACE–decidability of Japaridze's polymodal logic. // Advances in Modal Logic. — 2008. — V. 7 (College Publications, London). — P. 289–304.

[8] E. V. Dashkov, 'On the positive fragment of the polymodal provability logic GLP', *Matematicheskie Zametki*, 91(3):331–346, 2012. English translation: *Mathematical Notes*, 91(3):318–333, 2012.

[9] L. D. Beklemishev, 'Positive provability logic for uniform reflection principles', *Annals of Pure and Applied Logic*, 165:82–105, 2014.

[10] *Beklemishev L. D.* On the reflection calculus with partial conservativity operators. // Lecture Notes in Computer Science. — 2017. — V. 10388. — P. 48–67.

[11] *Beklemishev L. D.* A universal algebra for the variable-free fragment of RC^{∇}. // Symposium on Logical Foundations of Computer Science, S. Artemov and A. Nerode, editors, Lecture Notes in Computer Science. — 2018.

[12] *Pakhomov F., Walsh J.* Reducing ω-model reflection to iterated syntactic reflection. // Journal of Mathematical Logic. — 2023. — V. 23.

[13] *Joosten J. J.* Turing–Taylor expansions of arithmetical theories. // Stud. Logica 104, 1225–1243 (2015). https://doi.org/10.1007/s11225-016-9674-z

[14] *Fernandez-Duque, Joosten J. J.* Models of transfinite provability logic. // J. Symbol. Logic 78 (2), 543– 561 (2013).

[15] *Hermo Reyes E., Joosten J. J.* The logic of Turing progressions. // ArXiv:1604.08705v2 [math.LO], 2017

[16] *Borges A., Joosten J. J.* Strictly Positive Fragments of the Provability Logic of Heyting Arithmetic. // https://doi.org/10.48550/arXiv.2312.14727

[17] *Kurucz A., Wolter F., Zakharyaschev M.* Islands of Tractability for Relational Constraints: Towards Dichotomy Results for the Description of Logic EL. // Advances in Modal Logic. — 1998. — P. 271–291.

[18] *Gabbay D., Kurucz A., Wolter F., Zakharyaschev M.* Many-Dimensional Modal Logics: Theory and Applications. // Studies in Logic and the Foundations of Mathematics. — 2003. 2003.

[19] *Kurucz A., Wolter F., Zakharyaschev M.* Islands of tractability for relational constraints: towards dichotomy results for the description logic EL. // London, UK: College Publications. — 2010. — P. 271–291.

[20] *Kikot S., Kurucz A., Tanaka Y., Wolter F., Zakharyaschev M.* On the completeness of EL-equations: First results. // 11th International Conference on Advances in Modal Logic, Short Papers (Budapest, 30 August-2 September 2016). — P. 82–87.

[21] *Kikot S., Kurucz A., Wolter F., Zakharyaschev M.* On strictly positive modal logics with S4.3 frames. // *Advances in Modal Logic, Vol.12*

(eds.: G.Bezhanishvili, G.D'Agostino, G.Metcalfe and T.Studer), College Publications (2018), pp.399-418.

[22] *Kikot S., Kurucz A., Tanaka Y., Wolter F., Zakharyaschev M.* Kripke completeness of strictly positive modal logics over meet-semilattices with operators. // *Journal of Symbolic Logic*, vol.84(2019), pp.533-588.

[23] *Kikot S., Kudinov A.* On Strictly Positive Fragments of Modal Logics with Confluence. // Mathematics 2022, 10, 3701.

[24] L. D. Beklemishev, 'A note on strictly positive logics and word rewriting systems', In S. ODINTSOV, Larisa Maksimova, editors, on Implication, Interpolation, and Definability. Outstanding Contributions to Logic series, v. 15, Springer, 2017, pp. 61–70. Preprint ArXiv:1509.00666, 2015.

[25] L. L. Maksimova, V. V. Rybakov, 'Lattices of modal logics', *Algebra I Logika*, 13:105–122, 1974.

[26] W. J. Blok, 'Varieties of interior algebras'. Doctoral thesis, 1976.

[27] Scroggs, S. 'Extensions of the Lewis system S5', *The Journal of Symbolic Logic*, v. 16(2), 1951, pp. 112–120.

[28] A. CHAGROV, M. ZAKHARYASCHEV, *Modal Logic*, Oxford University Press, 1997.

[29] *Kruskal J. B.* The theory of well quasi-ordering: a frequently discovered concept // J. Combin. Theory. — 1972. — Ser. A 13. — P. 297–305.

[30] *Nash-Williams C.St.J.A.* On well-quasi-ordering finite trees. // Proceedings of Cambridge Phil.Soc 59. — 1963. — P. 833–835.

[31] *Laver R.* On Fraisse's order type conjecture. // Annals of Mathematics. — 1971. — V. 93. — P. 89–111.

[32] *Dershowitz N., Manna Z.* Proving termination with multiset orderings. // Lecture Notes in Computer Science. — 1979. — V. 71. — P. 188–202.

[33] *Egon Borger, Erich Gradel, Yuri Gurevich.* The Classical Decision Problem. — 1997.

[34] *Simpson S. G.* Nichtbeweisbarkeit von gewissen kombinatorischen Eigenschaften endlicher Baume. // Arch. Math. Logik Grundlag. — 1985. — V. 25 (1). — P. 45–65.

[35] *Gallier J.* What's so special about Kruskal's theorem and the ordinal Γ_0? A survey of some results in proof theory. // Annals of Pure and Applied Logic. — 1991. — V. 53. — P. 199–260.

[36] *Carlucci L.* Worms, gaps, and hydras. // Mathematical Logic Quarterly. — 2005. — V. 51 (4). — P. 342–350.

[37] *Schuster P., Seisenberger M., Weiermann A.* (editors) Well-Quasi Orders in Computation, Logic, Language and Reasoning. // Trends in Logic. — 2020. — V. 53.

[38] *Dzamonja M., Schmitz S., Schnoebelen P.* On Ordinal Invariants in Well Quasi Orders and Finite Antichain Orders. // Well-Quasi Orders in Computation, Logic, Language and Reasoning. — 2020. — Trends in Logic, V. 53 (Schuster P., Seisenberger M., Weiermann A. editors), Springer, Cham. — P. 29–54.

[39] *D. H. J. de Jongh, Parikh R.* Well-partial orderings and hierarchies. // Nederl. Akad. Wetensch. — 1977. — Proc. Ser. A 80 = Indag. Math. 39, N. 3. — P. 195–207.

[40] S. Kaniskin. About positive fragments of modal logics. Thesis, Moscow State University, 2013 http://lpcs.math.msu.su/ zolin/msc/

[41] *Rathjen M., J. Van der Meeren, Weiermann A.* Well-partial-orderings and the big Veblen number. // Archive for Mathematical Logic. — 2015. — V. 54. — P. 193–230.

Author's publications on the topic of the dissertation

[42] M. V. Svyatlovskiy. Axiomatization and polynomial solvability of strictly positive fragments of certain modal logics // Math. Notes, 103:6 (2018), c. 952–967

[43] Svyatlovskiy, M. Modal Companions of K4+. Studia Logica 110 (2022), c. 1327–1347

[44] M. V. Svyatlovskiy. Well-quasi order on strictly positive formulas // Studies of MIPT. T.14, №3 (55), 129–138. 2022